TRUTH OPERATIONS AND LOGICAL-MATHEMATICAL RECURSIVITY ON THE PROPOSITIONAL CALCULUS BASIS OF THE *TRACTATUS* OF L. WITTGENSTEIN

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Abstract: The objective of this paper is to present the truth tables method of the propositional calculus of *Tractatus Logico-Philosophicus* as a result of computational procedures involving recursive operations in mathematics, since the secondary literature that is involved with such a problem fails to demonstrate such aspect of the work. The proposal is to demonstrate the base calculation of the truth operations as a consequence of the application of mathematical resources that involve the notion of recursivity, inspired both in the natural numbers as well as in the factorial calculus, as well as in the procedures recommended by the combinatorial analysis and the calculation of probability. It is hoped, therefore, to present the truth operations of the propositional calculus as coming from the application of arithmetic to the logical resources.
Keywords: Operations of truth, propositional calculus, recursivity, Logic, Mathematics, *Tractatus*, Wittgenstein.

Resumo: O objetivo do artigo é apresentar o método de tabelas de verdade do cálculo proposicional do Tractatus Logico-Philosophicus como decorrente dos procedimentos de cálculo que envolvem operações recursivas no âmbito da matemática, já que a literatura secundária que se envolve com tal problema é falha em demonstrar tal aspecto da obra. A proposta é demonstrar o cálculo de base das operações de verdade como consequência da aplicação de recursos matemáticos que envolvem a noção de recursividade, inspirada tanto no conjunto dos números naturais quanto no cálculo fatorial, bem como nos procedimentos preconizados pela análise combinatória e pelo cálculo de probabilidade. Espera-se, com isso, apresentar as operações de verdade do cálculo proposicional como provenientes da aplicação da aritmética aos recursos da lógica.
Palavras-chave: Operações de verdade, cálculo proposicional, recursividade, Lógica, Matemática Tractatus, Wittgenstein.

I

It's known that the tables, nowadays named truth tables, were conceived in an independent form by Wittgenstein and Emil Leon Post in 1921. The vast influence of *Tractatus* and the immediate recognition of the philosophy of Wittgenstein ended up associating the tables to the Austrian philosopher, therefore, overshadowing the role of Post in its creation. Wittgenstein used the tables to classify truth functions in a series. His intention was to define the validity of a complex argument starting based on the validity of its elementary propositions (TLP 4.27-4.45; 5.101). It was an attempt to construct an ideal notation capable of revealing the underlying logical syntax to any possible language. Such method expresses intuitively what it means to say that the propositional meaning is determined by combinations of truth. These combinations exhibit the function of the logical operators in such a way that nothing is added by them. Logical operators *do not replace* anything because they are only elements of "copulation", a "kind of cement" that serves to bind the material components of propositions and that remains after the abstraction of such components[1]. From the elimination resulted certain conditions of truth from the truth possibilities expressed by the propositional bipolarity. All truth possibilities are determined in logical space in order to safeguard the determinability of propositional meaning.

The truth tables advocate that given *n* elementary propositions, there are 2^n possibilities or possible combinations of their truth values: 2^n possibilities of realization or non-realization of what they enunciate - being 2 of their truth possibilities (T or F) and *n* the number of molecular propositions. The representation that a fundamental of truth happens (that the result of the operation is T), which already contains the representation of what does not occur, can be made visible by the truth operation in propositional calculus.

For illustrative purposes, look at the result of the material implication that is presented in TLP 4,442:

'*p*	*q*	'
T	T	T
F	T	T
T	F	
F	F	T.

[1] It was said by Wittgenstein on the Letters to Russell, summer 1912 – 1.13, that are reunited at: WRIGHT, G. H. von (org.). *Letters to Russell, Keynes and Moore*. B. F. McGuinness. Oxford: Blackwell, 1974.

Even if the table was not completely filled in the example given by the *Tractatus*, it is known from the presentation of the "propositional sign", "(TTFT) (*p, q*)" (TLP 4.442), that it is about the truth function "*p⊃q*". In this case, "*p⊃q*" has three fundamentals of truth [(VV), (FT), (FF)] and its truth conditions are (TTFT).

The particularity of the truth tables proposed by Wittgenstein in relation to the models we find in the current logic manuals is that they appear in *Tractatus* in quotation marks and do not present the proposition at the top of the right column. This means that: i) they do not define propositional connectives, ii) they do not specify the truth conditions of molecular propositions, and iii) they are propositional signs that express molecular propositions without resorting to constants (as Wittgenstein calls it), or connectives or logical operators. What establishes a truth table as a propositional sign is the set of rules by which the spatial arrangement of its constituents refers to the set of internal relations between the various possibilities, elementary and molecular, involved in the definition of the propositional sense. These rules define the syntactic structure of the table as a symbol and it is such structure that symbolizes, and everything else is logically irrelevant. The table is the internal structure of the molecular possibility itself, the realization of which it represents and, to that extent, once made a propositional sign, is a logical figuration of its meaning.

Truth tables emerge from the evolution of Wittgenstein's thinking while seeking an alternative to Frege and Russell's truth-functional notation, since he identified that in such notation, sentences that had been treated as distinct constitute a single symbol. Frege and Russell presented alternative ways of writing the same proposition[2]. The first path proposed by Frege and Russell's truth-functional notation was "*ab* notation".

In *Notes on Logic* (NL), Wittgenstein says what such notation consists of:

> As the ab (TF)-functions of atomic propositions are bi-polar propositions again, we can perform ab operations on them. Then two letters T and F to these poles. In this notation all that matters in the correlation of the outside poles to the poles of the atomic propositions. (NL, 1913, p. 101).

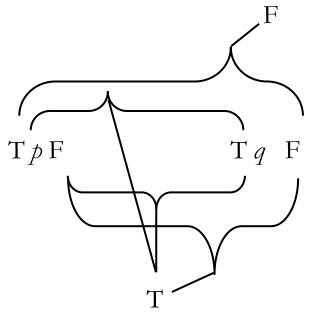
This is possible in the following way: a proposition "*p*" is written as "a-*p*-b" and its negation, i.e., "~*p*" is written as "b-a-*p*-b-a", being that *a* and *b* are two poles from the proposition. The symbol "a-b-a-*p*-b-a-b" is "~~*p*", that is identical to "a-*p*-b", i.e, "*p*". According to Wittgenstein, "It is therefore possible to construct all possible ab-functions by performing one ab-operation repeatedly, and we can therefore talk of all ab-functions as of all those functions which can be obtained by performing this ab-operation repeatedly" (NL, 1913, p. 102).

Wittgenstein sought to extend the application of *ab* notation to quantifiers.

> The application of ab notation to apparent variable propositions becomes clear if we consider that, for example, the proposition "for all x, ϕx" is to be true when ϕx is true for all x's and false when ϕx is false for some x's. We see that *some* and *all* occur simultaneously in the proper apparent variable notation. The notation is
>
> The notation is:
> to (x)ϕx: a-(x)-a-ϕx-b-(∃x)-b
> to (∃x)ϕx: a-(∃x)-a-ϕx-b-(x)-b
> Old definitions now become tautologous (NL, 1913, p. 103).

As it turns out, "a-(x)-a-ϕx-b-(∃x)-b" corresponds to "(x) ϕx" and "a-(∃x)-a-ϕx-b-(x) -b" corresponds to "(∃x) ϕx". The notation of internal negation is given by the reversal of the internal ab poles, thus symbolized ("(x) ~ ϕx"). And the notation of external negation by the reversal of the outer ab poles, thus symbolized ("~ (x) ϕx"). The symbolization for quantifiers shows that the argument's quantifiers are meaningful propositions, since they have two poles, not first-level function names, as Frege wanted. This is why "the old definitions now become tautologous".

The second path proposed by Wittgenstein to Frege and Russell's truth-functional notation was a two-dimensional variation of the *ab* variation that, according to him, can display the connections between the poles of molecular and atomic propositions that constitute them. This procedure is presented in a letter addressed to Russell between November and December 1913. According to Wittgenstein, it is a procedure that distinguishes tautologies, contradictions, and contingencies; it is taken up in aphorism 6.1203 of the *Tractatus*. Here is how to represent, through this device, the material implication ("*p⊃q*") of TLP 4.442, presented above:



[2] This matter will not be discussed in details once it is not the objective of this text.

According to Wittgenstein, the notation is the one that follows: "T *p* F" and "T *q* F" is used instead of "*p*" and "*q*". Through the braces are expressed the combinations of truth and through the traces the coordination of the truthness or falsehood of the proposition as a whole to the combinations of truth. With this procedure it is possible to recognize a tautology as such (TLP 6.1203).

In the evolutionary process of propositional calculus proposed by Wittgenstein, the *ab* notation eventually leads to truth tables (TLP 4.27-4.45; 5.101), which are nothing more than a truth operation using the procedure of calculating a formal series, whose recursive character ensures that linguistic signals can be repeatedly applied to a result (TLP 5.23).

The understanding that underlies the notion of propositional calculus requires understanding the rules of use of the five logical operators, presented here as in the current logic manuals, which are negation (~), conjunction (∧), disjunction (∨), material implication (→) and biconditional (↔). The purpose of introducing the calculation of these operators here is the need to demonstrate that their application is part of a formal series calculation procedure using two formulas given by Wittgenstein in *Tractatus* and which will be discussed ahead. The use of operators proposes to account for how a complex proposition is formed from elementary propositions, without an increase in terms of representational content to the resulting complex proposition, since each operator's sign does not mean objects, but the application of an operation.

Negation ("no") has its use extended to other modes of expression, such as "it is not true that", "it is false that" or by prefixes such as "un", "a-", etc. The negation of every true statement is false. Thus, in a sentence such as "Ruth Barcan is a philosopher" (*p*), we can form her negation by saying "Ruth Barcan is not a philosopher" (~*p*), which can be represented by a very simple truth table.

p	**~*p***
T	F
F	T

Unlike the belief that denying a proposition is merely to make an operation on it, that is, to construct one proposition from another (TLP 5.23), a negation is an operation applied to a propositional structure whose result is a complex proposition; so, it is a real operation. The result of this operation is a truth function, whose determination of its truth value is given from the determination of the truth value of its component proposition (TLP 5.234). The application of this operation involves a formal property relative to the applied rule, which is the application of the rule itself, as it does not involve combinatorial possibilities between names. This operation has a definite meaning, since it always gives the basis for the result and has a recursive character that generates a formal series: "*p*", "~*p*" "~~*p*", "~~~*p*", "~~~~*p*".

For the second case, we have an operation that involves more than one elementary proposition. In it, two *conjunctives* combine resulting in a *conjunction*. The result of a conjunction will be true when its antecedent and consequent are true at the same time; otherwise it will be false. In the sentence "Ruth Barcan is a philosopher and Ruth Barcan is a mathematician", replacing the elementary propositions with the constants of individual *p* and *q*, we have from the application of formula 2^n four truth values. Representing the truth possibilities of a conjunction by the truth values of its conjunctives, we have the following truth table:

p	***q***	***p*∧*q***
T	T	T
F	T	F
T	F	F
F	F	F

Since the operations have a recursive character, a conjunction can be applied again to obtain a new result. In every operation whose character is recursive, the recursivity is absorbed by all elements of the resulting operation. Thus, the conjunction, being also recursive, has *commutative properties* ((*p*∧*q*) ↔ (*q*∧*p*)), *associative properties* ((*p*∧*q*) ∧ *r*) ↔ (*p* ∧ (*q*∧*r*)), *distributive properties*. (*p* ∧ (*qvr*)) ↔ ((*p*∧*q*) ∨ (*p*∧*r*)), in addition to being involved with *De Morgan's Laws* ~(*p*∧*q*) ↔ (~*p*∨~*q*) e ~(*p*∨*q*) ↔ (~*p*∧~*q*).

In the *disjunction* (or alternation) operation, the disjunctives are operated using the "or", which is symbolized by the "∨" operator placed between two elementary propositions. A disjunction is false only if both disjunctives are false.

Considering two forms of interpretation, disjunction can be conceived as *inclusive* and *exclusive*. In the first form, it can be said that "an inclusive disjunction is true if either disjunctive or both are true; only if both are false will inclusive disjunction be false. The 'or' inclusive has sense of 'one or the other, possibly both'" (COPI, 1978, p. 229). – "'It's either raining or shining', we usually have one thing or another, but sometimes we happen to have both at the same time". In the second form, exclusive disjunction states that at least one of the statements is true, but not both are true, that is, either one thing or the other. For example, either "Leibniz is the inventor of calculus or Newton is the inventor of calculus". In exclusive disjunction, one alternative must exclude the other. If both disjunctives are true, the disjunction will be false.

Propositional calculus deals with inclusive disjunction[3]. Because of this, the result of "Leibniz is the inventor of calculus or Newton is the inventor of calculus," whether both elementary propositions are true or only one is truth, will always be true. The result will only be false if both elementary propositions are false, as represented by the following table:

p	q	$p \vee q$
T	T	T
F	T	T
T	F	T
F	F	F

The recursive character of disjunction manifests itself in formulas such as *idempotence of disjunction* $(p \vee p) \leftrightarrow p$, *commutativity of disjunction* $(p \vee q) \leftrightarrow (q \vee p)$, *associativity of disjunction* $(p \vee (q \vee r)) \leftrightarrow ((p \vee q) \vee r)$ and *disjunctive syllogism* $((p \vee q) \wedge {\sim}p) \rightarrow q$.

Material implication is another operation of propositional calculus and is known as *conditional operation* ("if *p*, then *q*"), represented by the operator "→" or "⊃". The operation of the material implication presupposes that the result of the calculation will be false only if the antecedent is true and the consequent is false, which seems obvious, since it is to be expected that if we state that *p* implies *q* that we deny the possibility that *p* is true and *q* is false. The result of the operation of a material implication by means of a truth table is as follows:

p	q	$p \rightarrow q$
T	T	T
F	T	T
T	F	F
F	F	T

A conditional is true under those value assignments to its elementary propositions where its antecedent is false and its consequent is true or false, and under those where its consequent is true and its antecedent is true or false. The conditional $p \rightarrow q$ has three truth foundations, that is, it is true under these value assignments: in the first, second, and last lines of values in the table above.

The material implication raises interesting problems for logic related to understanding the semantic content of the statements of the antecedent, the consequent and the result of the operation. It seems counterintuitive, for example, to conceive that an operation in which the antecedent and consequent are false has a true result. In saying, for example, "if the earth is square, then Mars is gaseous" we do not think we have obtained a legitimate result of a conditional operation, after all, in such an argument the conclusion does not seem to come from the premise, besides having two false assertions. However, it is said that the reason for classical logic to admit conditional operation as presented above is that it is more appropriate to work with it in mathematics. In fact, the solution to the conditional operation is quite simple: just admit that $p \rightarrow q$ is equivalent to ${\sim}(p \wedge {\sim}q)$, that is, we have $p \rightarrow q$ *if it is not possible for us to have true p and false q*. The conditional operation between two distinct elementary propositions is equivalent to the principle of non-contradiction.

The recursive character of material implication occurs, for example, in formulas such as the *counterposition* $(p \rightarrow q) \leftrightarrow ({\sim}q \rightarrow {\sim}p)$, *modus ponens* $(p \wedge (p \rightarrow q) \rightarrow q$, *modus tollens* $({\sim}q \wedge (p \rightarrow q)) \rightarrow {\sim}p$, *hypothetical syllogism* $((p \vee q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$, *Peirce's law* $((p \rightarrow q) \rightarrow p) \rightarrow p$ *and Duns Scot's law* ${\sim}p \rightarrow (p \rightarrow q)$.

The *bi-conditional*, on the other hand, represented by the equivalence operator (↔), corresponds to implication in both directions: $p \rightarrow q$ and $q \rightarrow p$. The statement of this operation is "if and only if" and its result in the truth table is as follows:

p	q	$p \leftrightarrow q$
T	T	T
F	T	F

[3] "The exclusive disjunction is not usually part of first-order logic systems, since a proposition $p \vee q$ is strictly equivalent to $p \leftrightarrow {\sim}q$" (BRANQUINHO; MURCHO; GOMES, 2006, p. 264).

T	F	F
F	F	T

If *p* is equivalent to *q*, they have the same value and the result of the truth table operation should have T in the lines where the truth values are equal, that is, in the first and fourth lines. In other cases, since the values are different and, therefore, not equivalent, the value will be F. The recursive character of equivalence can be expressed in operations such as *associativity equivalence* ($p \leftrightarrow (q \leftrightarrow r)) \leftrightarrow ((p \leftrightarrow q)) \leftrightarrow r$) and *commutability equivalence* ($p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p$).

The result of a truth operation by a table is intuitive: given the combination of values of the elementary propositions, it is possible to infer the value of the complex propositions by the use of operators that connect them. In the operation, separating the elementary propositions and delegating to them an individual constant, a symbol that represents them, is enough, as presented earlier. Given the symbolization of elementary propositions by individual constants and being aware of how to use the connectives, formula 2^n is employed to have the possibilities of truth values from propositions. Therefore, the possibilities of T and F are distributed according to the probability that they will combine to finally apply the calculus of operators for each proposition according to the rules mentioned above. The result of the operation will be ensured as valid when a single row of the table is all true. The individual constants (*p, q, r, s*) will be the table's truth arguments, the possibilities that they are T or F are the truth possibilities and the vertical column, result of the intended operation, presents the truth conditions that, in the end, can present a tautology, a contradiction or a contingency.

II

As seen before, each truth table is the result of applying the truth function operation, whose truth value determination is given from the truth value determination of its component proposition (TLP 5.234). As it is a truth operation, it means that its recursive character is assured due to the possibility of operations being repeatedly applied to a result. To guarantee the operation, it is necessary the elementary proposition to which the operation is applied for the first time - called the base of the operation - which may be composed of more propositions until it reaches the resulting proposition - the result of the operation. Since the base is present at both the beginning and the end of the operation, it is common to both (TLP 5.24). The fixation of the base occurs by direct enumeration "(*p*,*q*)", by specifying the function fx that acts on a set of propositions "$\{f\mathrm{a}, f\mathrm{b}, f\mathrm{c}, f\mathrm{d},...\}$" and by specifying a formal law (TLP 5.501).

In the case of truth tables, as we have a finite number of propositions, the base can be defined by direct enumeration. In the case of "$p \rightarrow q$", for example, the basis of the operation will be "(*p*,*q*)". Thus, "if '(*p*,*q*)' is a member of the series, '$p \rightarrow q$' is also a member of the series". One can also decide on specifying the function fx to determine the relationship between the elements of the pair $\langle x, y\rangle$. Given a set A and a set B, a function is the relationship between the elements of set A and the elements of set B, represented by f: A → B; $f(x) = y$, where $x \in$ A and $y \in$ B. It occurs that each element x of set A has only one correspondent to y in set B. If there is a relationship between the elements of A and B that does not comply with this premise, then you cannot call it a function. When all x elements of set A always have the same y of set B, the function is said to be constant. For cases where f: A → B; $f(x) = y$ and f: B → A; $f(y) = x$ occurs simultaneously, we have a bijector function; this means that there was a two-way correspondence there, as advocated by the *Tractatus* (TLP 3.21). In this case, the consequence of understanding the use of the fx base is that the application of f to some element $x \in$ A is represented by $f(x)$. Thus, we represent the fact that y is the image of x by the function f writing $f(x)=y$.

Both "$p \rightarrow q$" and its respective table are propositional signs. The truth tables, in this case, represent a "logically privileged" propositional sign, since they spatially dispose the internal structure of a molecular possibility, requiring nothing more than an indication of how the truth conditions of the proposition are defined in terms of the conditions of truth of the elementary propositions.

As it will be seen, such tables are the consequence of the application of two mathematical formulas proposed by Wittgenstein in the *Tractatus*, namely:

$$K_n = \sum_{v=0}^{n} \binom{n}{v} \text{ and } \sum_{\mathrm{k}=0}^{\mathrm{K}_n} \binom{\mathrm{K}_n}{\mathrm{k}} = L_n$$

The purpose of applying the equations in *Tractatus* are:

a) To calculate the fixed combinatorial number of possibilities of truth, the possibilities for *n* elementary propositions (corresponding to *n* state of affairs);
b) To calculate the number of molecular propositions from the number of possibilities of truth.

As for the first formula says Wittgenstein: "For *n* states of affairs, there are

$$K_n = \sum_{v=0}^{n} \binom{n}{v} \text{ possibilities of existence and non} - \text{existence" (TLP 4.27).}$$

To understand the language of the equation, it is necessary to identify each element that constitutes it and extract the underlying concept. It is a formula for calculating state of affairs, and the equation has four elements:

$$\mathrm{K}_n(1) = (2)\sum_{v=0}^{n}(3)\binom{n}{v}(4)$$

Element (1) represents the number of state of affairs within a set with *n* elementary propositions. Element (2) is the mathematical operator that equals both sides of the equation; this means that any changes that are made on one side must be made on the other. Element (3) is the mathematical operator that sums all the results of the function associated with it, obeying a range, which in this case is also the domain of this function. The domain should be understood here as a set of elements (or values) for which the function is valid. Element (4) is the simple combination function. The value of υ will vary within the range proposed by the sum and the formula is calculated repeatedly until all-natural numbers within the range are exhausted, i.e. until υ equals *n*. Each of the (*n*+1) results of the combination will be summed by operator (3) over the entire given range and associated via operator (2) to element (1).

> The number K_n means that, for n state of affairs, there are - if we calculate the given formula - 2^n possibilities for the distribution of obtaining or not obtaining state of affairs. At the same time, this determines 2^n truth possibilities of the corresponding combination of being true or false (PILCH, 2017, p. 30).

This can be demonstrated as it follows. Let us use binomial coefficient or binomial number notation to formally represent the simple combination:

$$C_{n,v} = \binom{n}{v}$$

Given a *Sample Space* S = {*p*, *q*, *r*}, the various possible combinations with the events be contained, without repetition of events, are eight (8). Each is a subset of S: { }; {*p*}; {*q*}; {*r*}; {*p*, *q*}; {*p*, *r*}; {*q*, *r*}; {*p*, *q*, *r*}. Each of these subsets is a state of the *Sample Space* S events. The possibility of *n*=3 distinct and independent state of affairs or events is the number of subsets of the *Sample Space*. The eight (8) combinations are distributed according to their state, characterized by the variable υ of the formula, that is, each combination is the count of subsets that can be in this state υ, as we will demonstrate below: 1 (one) combination, or state, where no event occurs (*n*=3 and υ=0); 3 (three) states in which only one of the events occurs (*n*=3 and υ=1), 3 (three) possibilities of any pair of events occurring (*n*=3 and υ=2) and; 1 (one) where all three events occur (*n*=3 and υ=3), according to the model below.

Quantity of Events	Sample Space $\boldsymbol{S}$	Combinations $\binom{n}{v}$, where: $0 \le v \le n$		Value of n		Summation $\sum_{v=0}^{n}\binom{n}{v}$	Value K_n
To *n*=0	{ }	$\binom{0}{0}$	1	***0***		1	1
To *n*=1	{*p*}	$\binom{1}{0}\binom{1}{1}$	1	***1***		1+1	2
To *n*=2	{*p*, *q*}	$\binom{2}{0}\binom{2}{1}\binom{2}{2}$	1	***2***	1	1+2+1	4
To *n*=3	{*p*, *q*, *r*}	$\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$	1	***3***	3 1	1+3+3+1	8
To *n*=4	{*p*, *q*, *r*, *s*}	$\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4}$	1	***4***	6 4 1	1+4+6+4+1	16
To *n*=5	{*p*, *q*, *r*, *s*, *t*}	$\binom{5}{0}\binom{5}{1}\binom{5}{2}\binom{5}{3}\binom{5}{4}\binom{5}{5}$	1	***5***	10 10 5 1	1+5+10+10+5+1	32
To *n*	{*p*, *q*, *r*, *s*,...}	$\binom{n}{0}\binom{n}{1}\binom{n}{2}\binom{n}{...}\binom{n}{...}\binom{n}{n-1}\binom{n}{n}$	1	$\boldsymbol{n}$	... ... *n* 1	1+*n*+...+...+...+*n* +1	$\boldsymbol{2^n}$

Whenever we have a combination where υ=0, we have only one empty set, which validates the recursion of the combination in relation to the concept of 0! = 1, which makes it possible to calculate the combination $C_{n,0} = \binom{n}{0}$ which, in this case, always will be one (1), defined like this even when *n*=0.

The summation of each number row in the table above is a function of the number of elements in the set S, i.e. it is a function of *n*. So: *n*=0, $\sum = 1$; *n*=1, $\sum = 2$; *n*=2, $\sum = 4$; *n*=3, $\sum = 8$; *n*=4, $\sum = 16$, and so on. Using ordered pairs (n, K_n) to determine the function that sum each row, we have:

$$f(n) = K_n \qquad f(2) = 4$$
$$f(0) = 1 \qquad f(3) = 8$$
$$f(1) = 2 \qquad f(4) = 1$$

Where:

n = the number of elementary propositions;

K_n = summation of the results of the coefficients of row *n*.

Notice that the recursion of the functions, due to the possibility of repeated application of the function *f* (0) = 1:

a) *f*(3) = 2x*f* (2), where *f*(2) = 2x*f*(1);

b) *f*(3) = 2x2x*f* (1), where *f*(1) = 2x*f*(0);

c) *f*(3) = 2x2x2x*f* (0), where *f*(0) = 1.

Truth functions, in this case, are the counterface of exponential functions in mathematics. Look at the example of the argument ((*p*^*q*) → r). Here's your truth table:

X_1 *p*	X_2 *q*	X_3 *r*	X_4 $((p \wedge q) \rightarrow r)$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

For the argument ((*p*^*q*) → *r*), we have the function *f* defined by $X_4 = f(X_1, X_2, X_3)$ as it follows $(X_1, X_2, X_3) \in \{T,F\}^3 \rightarrow X_4 = f(X_1, X_2, X_3) \in \{T,F\}$ with the following description of the truth table:

f(TTT) = T   *f*(FTT) = T

f(TTF) = F   *f*(FTF) = T

f(TFT) = T   *f*(FFT) = T

f(TFF) = T   *f*(FFF) = T

The exponentiation 2^n is evidenced by the applied procedure, that is, if we have *f*(3), we see that 2x2x2 = 8. This means that the number of multiplications of factor 2 is equal to the number of elementary propositions *n*:

$d)\ f(3) = 2^3 = 8;$

$e) \Rightarrow f(n) = 2^n$

$f)\ K_n = 2^n$

Which results in the following formula (TLP 4.27):

$$K_n = f(n) = \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} = 2^n, \{n, K_n, \upsilon\} \in N$$

For the second formula which says that "for *n* elementary propositions there are $\sum_{k=0}^{K_n} \binom{K_n}{k} = L_n$ ways in which a proposition can agree and disagree with their truth-possibilities" (TLP 4.42), the understanding is analogous, but the extension, the size or quantity of propositions is given by K_n, as opposed to the previous formula where it was given by *n*. K_n and *n* are also the upper range of the summation. The result of the second equation is derived from the result of the first, that is, the value of one is directly associated with the value of the other as an ordered pair, whenever K_n is x_1, L_n will be y_1. The notation of this mathematical formula, therefore, is as it follows:

> K_n = number of state of affairs of a given set with *n* elementary propositions. The first formula calculates the value of K_n that will be used recursively for the calculation of the second, which calculates the value of L_n;
> L_n = amount of truth possibilities within a set with K_n state of affairs. The value of L_n is derived from K_n, which in its turn is derived from *n*. It is observed that the recursion in relation to *n*, that is, it is possible to write everything as a function of *n* only;
> $\sum_i^f \binom{n}{k}$= mathematical notation used to calculate the summation of the results of $\binom{n}{k}$, with *k* ranging from *i* to *f*;
> $\binom{n}{k}$ = Mathematical notation used to calculate the number of simple combinations of *n* elements taken *k* to *k*. With: $0 \leq k \leq n$.

Of the four elements of the equations, *one* is the value you want to calculate, *two* are the operators, and *one* is a function. The combination function counts the number of occurrences of a combination of elements of a given finite set; in this case, a set of elementary propositions. But it can be any enumerable or countable set, that is, of a set in which each element can be associated with an element of natural numbers; a set from which its elements can be enumerated, counted and ordered.

Since a function of combinatorial analysis (a simple combination) is used, we have to understand what is the relationship between the state of affairs represented by the elementary propositions $\{p, q, r, ...\}$, with the numerical values $\{1, 2, 3, ...\}$ used for the calculus of formula. The numerical result of Wittgenstein's formulas is the collections of combinations of conditions of existence (state of affairs) and combinations of conditions of truth (agreement and disagreement of a proposition with the possibilities of truth) from the conditions of existence. When calculating the formulas, we obtain the count of: i) all conditions of existence of the elementary propositions $\{p, q, r, ...\}$, that is, the space of states[4], where a world model of *n Sachverhalte* (states of affair) implies 2^n possible distributions of each assignment; ii) all circumstances in which conditions of existence may or may not be true (or false), that is, the propositional space where the number of molecular propositions is calculated by the number of possibilities of truth - 2^{2^n} molecular propositions by *n* elementary propositions. This numerical procedure allows to map or mathematically distribute any condition of truth from the science of any condition of existence of elementary propositions. In both cases, it is a finite count of conditions from a given finite set of carefully ordered elementary or molecular propositions.

The combinatorial analysis is widely used by probability and logic scholars, as it allows working with groups of events that are directly related to counting, and Wittgenstein works with this feature, which he knew very well as a mechanical engineer, to calculate the conditions of possibility of existence and nonexistence of states of affairs, as well as to deal with the agreement and disagreement of a proposition with the truth possibilities of n elementary propositions.

> When we consider simple combinations of *n* elements taken *p* through *p*, we have groupings of *p* elements, taken from the *n* available elements, which differ only in the nature of the elements, that is, it matters only who participates in the group. (SANTOS; MELLO; MURARI, 2007, p. 62-63).

The set of possible outcomes of a stage of a random event or phenomenon, that is, the set of different outcomes that can occur each time the stage is launched, is given by $S=\{r_1, r_2, r_3, ..., r_n\}$, which we will call *Sample Space* S[5], where the index *n* is equal to the number of elements in set S. Each element in set S then corresponds to one and only one of the possible outcomes of the event (or state of affairs, or fact)[6] or random phenomenon.

It is important to reiterate that from a set of atomic facts, or events, each fact is an event (or state of affairs). Tractarian facts are based on the eternal objects of the work, objects which are linked to the genuine names of language and which, articulated, form the elementary propositions, which are the figuration of reality (TLP 4.01). This relationship between logic and ontology serves to guarantee figuration as well as determinability of propositional meaning, since the projective relation procedure is punctual ("One name stands for one thing,

[4] We propose here a notion of Martin Pilch (2017), who, to understand the structure of the logical space of the *Tractatus*, suggests a scheme that divides such space into three: parameter space, state space and propositional space. It is from the understanding of how such spaces work that we propose to understand the Wittgensteinian notion of logical space as a whole.
[5] Equivalent here to the parameter space of Pilch (2017).
[6] The treatment given to the term "event" here is the same as "state of affairs" or "fact".

another for another thing, and they are combined with one another. In this way the whole group - like a tableau vivant - presents a state of affairs"- TLP 4.0311).

The tractarian notions of "atomic" and "molecular" are also mathematical. Mathematics assumes that all subsets of a finite *Sample Space* S are also events, and the subset that has only one element of the set S is an atomic event. When there is more than one element in this subset, you have a compound or molecular event. The empty subset { }, which is a subset of all sets, can be understood as an empty event or where there was no event at all.

The table below will elucidate a binary mathematical event and how to count and combine its results. This is the event of playing two coins at the same time, one 1-libra coin and one 1-dollar coin, which from then on will be represented by the symbols *p* and *q* respectively. When playing the two coins, for the possible outcomes of occurrence of "Heads" we will signal with T. When the fact does not happen, that is, when it gives "Tails", we will signal with F.

Event: Throwing 2 Coins	
Occurrence of Heads	
p	***q***
T	T
T	F
F	T
F	F

So, we have a total count of 4 conditions of possibility of occurrence of "Heads" when playing both coins simultaneously. In the truth table language, employing the formula 2^n, we have 4 possibilities of the corresponding combination of these state of affairs being true or false.

In moving the analysis from this scope to that of the proposition, let us take the molecular proposition "Ruth Barcan is a philosopher and Ruth Barcan is a mathematician" as an example. We had already demonstrated, from the state space perspective, the truth table of conjunction. That table showed us four lines for the truth possibilities of states of affairs to be true or false (their possibilities of existence or non-existence) by applying formula 2^n. It was verified that this is a valid argument, by the first row of the table, whose contingency character was evidenced by the truth conditions, through the column of the result of the truth operation. Given a molecular proposition such as the one above, which allows us to calculate the chance that a particular specific random event will happen at this particular stage (eg, "**Ruth Barcan is a philosopher**" happens, in the context where "Ruth Barcan is a philosopher and Ruth Barcan is a mathematician") is the ratio between the size of the event result set that will be favorable and the set size of all possible event results.

For the example above, let us use the following notations:

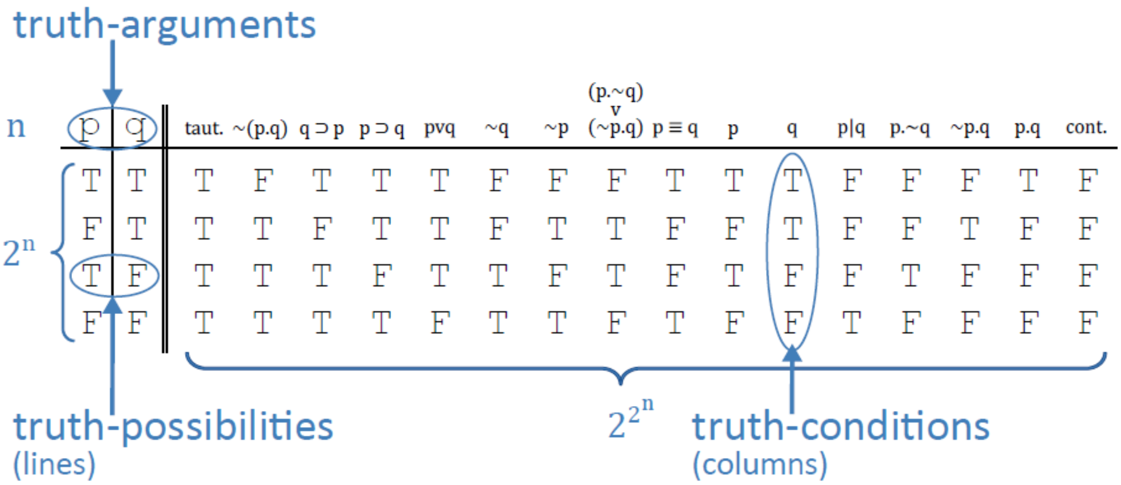
a) S = {*p*, *q*} = {(Ruth Barcan is a philosopher), (Ruth Barcan is a mathematician)} = Sample Space;
b) S_p = {(Ruth Barcan is a philosopher, Ruth Barcan is a mathematician), (Ruth Barcan is a philosopher, Ruth Barcan is **not** a mathematician), (Ruth Barcan is **not** a philosopher, Ruth Barcan is a mathematician), (Ruth Barcan is **not** a philosopher, Ruth Barcan is **not** a mathematician)} = all 4 (four) possible events = all subsets of S;
c) S_f = {(Ruth Barcan is a philosopher, Ruth Barcan is a mathematician), (Ruth Barcan is a philosopher, Ruth Barcan is **not** a mathematician)} = set of favorable events.

The probability (P) of a specific event to happen, for example, "Ruth Barcan is a philosopher" to happen, in the context of the above conjunction, is equal to the ratio between n_f (number of elements in the set S_f) and the number of elements n_p (number of elements of set S_p). Thus, the probability (P) of "Ruth Barcan is a philosopher" to happen (favorable event) is $n_f = 2$ divided by $n_p = 4$, i.e. P = 2/4, that is, *one* chance every *two* possibilities of truth or in 50% of the possibilities.

Just as it is possible to analyze in terms of probability the possibilities of "Heads" when throwing two coins, or to analyze that from a binary function such as *p.q* it is possible to extract its possibilities of truth, it is also possible to ascertain from the two truth arguments[7] *p* and *q*, what are the truth conditions of molecular propositions by *n* elementary propositions (2^{2^n}), using the second formula presented by Wittgenstein in aphorism 2.42 of the *Tractatus*.

[7] Wittgenstein calls the arguments of any truth function as "truth arguments." (TLP 5.01).

The formula $\sum_{k=0}^{K_n} \binom{K_n}{k} = L_n$ leads to 2^{2^n} molecular propositions by *n* elementary propositions. In *Tractatus* aphorism 5.101, Wittgenstein restricts a demonstration of the scheme resulting from the propositions by n = 2, which would give the following distribution picture of the binary function.



Ref. Pilch (2017, p. 31)

Wittgenstein's procedure in restricting the demonstration to *n*=2 has the following justification:

> The restriction to n = 2 is understandable because of the numbers' rapid increase resulting from the power of the double exponentiation, which leads for n = 4 to 16 truth-possibilities and 65636 different truth-conditions, but for n = 8 we have 2n = 256 with the number of 2^{2^n} possible truth-conditions growing to ≈ 1077 (PILCH, 2017, p. 31).

And the meaning of the application of the summation formulas of TLP 4.27 and TLP 4.42 could be expressed as a form of elucidation:

$\boldsymbol{n = 2}$
$\boldsymbol{K_n = 2^n}$
$\boldsymbol{2^2 = 4}$
$$\boldsymbol{L_n = \sum_{K=0}^{K_n} \binom{K_n}{K}}$$
$\boldsymbol{0 \leq K \leq K_n}$

Truth-possibilities of $\mathbf{K_n}$ = 4	Truth-argument 1 ***p***	Truth-argument 2 ***q***	**1** (p⊃p.q⊃q) Tautology	**2** (~(p.q))	**3** (q⊃p)	**4** (p⊃q)	**5** (p∨q)	**6** (~q)	**7** (~p)	**8** (p.~q) ∨ (q.~p)	**9** (p≡q)	**10** p	**11** q	**12** (pIq)	**13** (p.~q)	**14** (q.~p)	**15** (q.p)	**16** (p.~p.q.~q) Contradiction
1	T	T	T	F	T	T	T	F	F	F	T	T	T	F	F	F	T	F
2	F	T	T	T	F	T	T	F	T	T	F	F	T	F	F	T	F	F
3	T	F	T	T	T	F	T	T	F	T	F	T	F	F	T	F	F	F
4	F	F	T	T	T	T	F	T	T	F	T	F	F	T	F	F	F	F
Quantity of truth-conditions			1	4				6						4				1
Value of **K**			0	1				2						3				4
Combination			$\binom{4}{0}$	$\binom{4}{1}$				$\binom{4}{2}$						$\binom{4}{3}$				$\binom{4}{4}$

Truth-Conditions $\mathbf{L_n = 16}$

As it turns out, these are binary truth functions on which it is possible to calculate probabilistically both the possibility of existence and non-existence of *n* states of affairs (TLP 4.27), and the agreement and disagreement of a proposition with the possibilities of truth of *n* elementary propositions (TLP 4.42). For this, two formulas are essential to us: 2^n, with which it is possible to calculate the possibilities of existence or non-existence of *n* states of affairs, and 2^{2^n} with which it is possible to calculate the agreement and disagreement of a proposition with the possibilities of truth of *n* elementary propositions, from the truth arguments *p* and *q*. According to Pilch (2017, p. 31), this would even allow to represent the structured logical space in three similar sets: by the set of *n* elementary propositions, which will be truth arguments, representing the "parameter space"; by the set of possibilities of truth, which represents the "state space"; and by the set of this set, containing all possible molecular propositions, the "propositional space," as follows.

parameter space	state space	propositional space
n	2^n	2^{2^n}
truth-arguments	truth-possibilities	truth-conditions
states of affairs	situations (narrow sense)	situations (wider sense)
(elementary) facts	facts (narrow sense)	facts (wider sense)
elementary propositions	(molecular) propositions	all (molecular) propositions
elementary propositions	conjunctions of elementary propositions	arbitrary truth-functions of elementary propositions
set of all independent states of affairs	set of all possible distributions ("states") of the obtaining and not obtaining states of affairs	set of all subsets of alternative "states"

Ref. Pilch (2017, p. 31)

The numeric value of the binary function distribution table count above represents the enumerated collection of all possible combinations of propositions *p* and *q*.

The recursion procedure shown above allows one to create complex propositions in terms of simple propositions, previously defined propositions that will belong to the same class as the first. An example of this in mathematics is the set of natural numbers, which is defined from a single natural number (0) and carries the class property of natural numbers (0, 0 + 1, 0 + 1 + 1, etc.). Thus, each element of the set is derived from the first. If an element contains the property of the class, those derived from it will also contain it, thus forming what is called a recursively enumerable set. This recursion, which is an interesting property of the set of natural numbers, also becomes a property of truth operations when it becomes possible for a bijection to be made in order to make *Sample Space* S computable.

In the case of natural numbers, the criteria that constrain and define them are:

a) the number 0 is natural;
b) every natural number has a successor $n + 1$ and;
c) every successor is also natural as long as *n* is natural.

Only with $n_0 = 0$ is it possible to instantiate all elements of the set of natural numbers with cardinality *n*.

Just as the set of natural numbers is recursive, so is the factoring procedure in mathematics - and the elucidation of this here is of great relevance to the understanding of the application of summation formulas used by Wittgenstein in the *Tractatus*. When we define the factorization equation to calculate the numerical value of the factorial, we have the following formula:

$$(n + 1)! = n!.(n + 1), \forall n \in N \rightarrow 0! = 1$$

The definition above ($\mathbf{0! = 1}$) is the factor recursion criterion, just as the number 0 was for the natural number set. This definition makes the product between elements of any set S, with *n* elements consistent, once the empty product is clarified, which is nothing more than the result of the product of a factor by no number[8]. The empty product happens when we do a multiplication by no number, that is, when one of the factors is empty; in this case, the result of this product is the neutral element of multiplication, that is, a numerical value that does not change the value of the entire operation but is part of the factors of the operation as a whole.

In mathematics the number zero (0) can be a quantity of something, represent an empty set, or quantify nothingness. The factorial quantifies all permutations of *Sample Space* S. If *Sample Space* S has n=1 element, it has P_1=1 exchange. If it has n=2 elements, it has P_2=2 permutations. If it has n=3, it is already P_3=6. If n=0, *Sample Space* S = { } (empty) has an exchange P_0=1, equal to that of a sample space with only n=1 element.

In logic, mathematics or computing, a recursive language is understood to be a subset of a formal language. Formal language constitutes the *Sample Space* and each of the recursive languages constitutes the subsets of that space. The set S=$\{r_1, r_2, r_3, ..., r_n\}$ of possible outcomes of an event, which has n elements, is recursively enumerable because it is possible to associate each element of the set S with an element of the set of natural numbers N=$\{1, 2, ..., n\}$. The language of the factorial function enumerates all valid strings with n elements of the *sample space S*, and, as each element of the set S is associated with a natural number, by adding one more element to this set S, with the possible result $r_{(n+1)}$, there is no need to repeat the whole validation process, just from n add 1 more and check if there is already occurrence of this new validated chain of events. As it was said before, a subset of a recursive set is also recursive, and any result, language or procedure with recursive sets will also be recursive.

Concerning the recursion expressed in the *Tractatus* aphorism formula 4.42, we can say that there is a mapping or counting of the possible combinations of events that reflect the agreement and disagreement of a proposition with the truth possibilities of n propositions, or rather the truth possibilities of subsets of propositions within a *Sample Space* S. Among these 16 possibilities, presented in the table above, there is the possibility of no events occurring, as well as 2 or 3 of them and 1 or 2 not, for example.

If we take as an example the non-occurrence of states of affairs (hence their disagreement with reality, that is, the circumstance that the propositional calculation is false (F)), the number of events or event combinations where no F event occurs can be calculated by[9]:

$$C_{n,p} = \binom{n}{p}, where\ n = 4, p = 0$$

$$C_{4,0} = \binom{4}{0} = 1$$

$$C_{4,0} = 1$$

p	*q*	*(p→p).(q→q)*
T	T	T
F	T	T
T	F	T
F	F	T

There is 1 combination where no F event occurs. It is a tautology which in this case corresponds to the empty subset of *Sample Space* S - empty of F. That's why Wittgenstein states that "a tautology leaves open to reality the whole - the infinite whole - of logical space" (TLP 4.463). The same occurs with the empty set. The empty set is a subset of all sets, because from it you can compose all other subsets through enumerated recursion. In the example above, the empty subset allows that the space has all the equal possibilities of agreement.

8 The result of a product by 0 is always 0 (n.0 = 0) it must not be confused with the product of factor 0, because in this case the factor is a number.

9 It is noteworthy that in this case, as it is the agreement or disagreement of the proposition with two possibilities of truth, the value of n in the following calculations is equal to K_n.

The number of events, or event combinations, where only 1 false result (F) of the 4 events occurs can be calculated by:

$$C_{n,p} = \binom{n}{p}, where\ n = 4, p = 1$$

$$C_{4,1} = \binom{4}{1} = 4$$

$$C_{4,1} = 4$$

p	q	$\sim(p.q)$	$(q \rightarrow p)$	$(p \rightarrow q)$	$(p \vee q)$
T	T	F	T	T	T
F	T	T	F	T	T
T	F	T	T	F	T
F	F	T	T	T	F

Therefore, there are 4 combinations where only 1 F event occurs. Corresponds to the subsets with n=1 element that were presented by Wittgenstein in the above operations.

The number of events, or combinations of events, where 2 falsehoods (F) of the 4 events occur can be calculated by:

$$C_{n,p} = \binom{n}{p}, where\ n = 4, p = 2$$

$$C_{4,2} = \binom{4}{2} = 6$$

$$C_{4,2} = 6$$

p	q	$\sim q$	$\sim p$	$(p.\sim q) \vee (q.\sim p)$	$(p \leftrightarrow q)$	p	q
T	T	F	F	F	T	T	T
F	T	F	T	T	F	F	T
T	F	T	F	T	F	T	F
F	F	T	T	F	T	F	F

Here Wittgenstein presents 6 combinations where 2 false events (F) occur, as shown in the table on page 18. They correspond to subsets with n=2 elements.

The number of events, or event combinations, where 3 false results (F) of the 4 events occur can be calculated by:

$$C_{n,p} = \binom{n}{p}, where\ n = 4, p = 3$$

$$C_{4,3} = \binom{4}{3} = 4$$

$$C_{4,3} = 4$$

p	q	$(p\mathrm{I}q)$	$(p.\sim q)$	$(q.\sim p)$	$(q.p)$
T	T	F	F	F	T
F	T	F	F	T	F
T	F	F	T	F	F
F	F	T	F	F	F

We have here 4 combinations where 3 false events occur. They correspond to the subsets with *n*=3 elements. From this we get the idea of complementary events, that is, sets that have the same amount of combinations, but with the complementary elements, in this case, the complementary event of each of the 4 combinations of $C_{4,3} = 4$ will be an event of the 4 combinations of $C_{4,1} = 4$. Being set S={1, 2, 3, 4}, a combination would be set {1} and its complement {2, 3, 4}.

The number of events, or combinations of events, where 4 falsehoods (F) of the 4 events occur can be calculated by:

$$C_{n,p} = \binom{n}{p}, where\ n = 4, p = 4$$

$$C_{4,4} = \binom{4}{4} = 1$$

$$C_{4,4} = 1$$

p	*q*	(*p*.~*p*) . (*q*.~*q*)
T	T	F
F	T	F
T	F	F
F	F	F

There is 1 combination where the 4 events are all false. Here we have a contradiction that "fills the whole of logical space leaving no point of it for reality" (TLP 4.463). This is an impossible truth proposition, as it does not represent any possible situation. This operation corresponds to the subset with *n*=4 elements.

The events that will make up this *Sample Space* S', of cardinality n'=K_n, are each subset of *Sample Space* S, that is, if the set S is composed of n distinct and independent events, the set S' is composed of n'=K_n events. These events are not always atomic, that is, it will occur that elements of set S' have more than one of the events of set S.

Counting the subset count of the *Sample Space* S = {*p*, *q*}, where each subset represents a column of the truth table, for: $n = 2; K_n = 4$, we have:

$$C_{4,0} = 1; C_{4,1} = 4; C_{4,2} = 6; C_{4,3} = 4; C_{4,4} = 1$$

$$f(K_n) = \sum_{K=0}^{K_n} \binom{K_n}{k} = L_n = 2^{K_n} = 2^{2^n}$$

$$\sum_{K=0}^{K_n} \binom{K_n}{K} = \sum_{K=0}^{4} \binom{4}{K}$$

$$\sum_{K=0}^{4} \binom{4}{K} = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$$

$$\sum_{K=0}^{4} \binom{4}{K} = C_{4,0} + C_{4,1} + C_{4,2} + C_{4,3} + C_{4,4} \sum_{K=0}^{4} \binom{4}{\mathrm{K}} = 1 + 4 + 6 + 4 + 1 = 16$$

$$\sum_{K=0}^{K_n} \binom{K_n}{\mathrm{K}} = \sum_{K=0}^{4} \binom{4}{\mathrm{K}} = L_2 = 16$$

$$L_n = \sum_{p=0}^{x} \binom{\boldsymbol{x}}{\mathbf{p}} = 2^x = y, \{x, y, n\} \in N, x = K_n = n'$$

$$L_n = \sum_{p=0}^{K_n} \binom{\boldsymbol{K_n}}{\mathbf{p}} = 2^{n'} = 2^{K_n} = 2^{2^n}, \{n\} \in N$$

Given the large gap left by the interpreters of Wittgenstein's *Tractatus* in demonstrating the foundations of the propositional calculus of the work from the foundations of mathematics, what was meant here was to demonstrate that the truth table method is part of a common procedure in the calculation that is based on recursive mathematical operations. The proposal was to demonstrate that the truth operations have a recursive character due to the possibility of repeated application of the truth function, as happens in the case of the set of natural numbers, in the factorial calculation, in the combinatorial analysis procedures, as well as in the calculus of probability.

The application of the summation equations employed in *Tractatus* aphorisms 4.27 and 4.42 is the evidence we need to elucidate the question of the foundations of tractarian propositional calculus from the resources of arithmetic.

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